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FINAL REPORT

for

MINIMUM VARIANCE PRECISION TRACKING AND ORBIT PREDICTION PROGRAM

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FOREWORD

Minimum Variance Precision Tracking and Orbit Prediction Program

This report was prepared by Analytical Mechanics Associates, Inc. in partial fulfillment of contract NAS5-2535, for the Special Projects Branch, Theoretical Division, G. S. F. C., NASA.

The report contains an extension and modification of the Kalman Minimum Variance filter for application to an orbit determination program.

The analysis was carried out by Samuel Pines and Henry Wolf of AMA with active assistance from R. K. Squires and D. Woolston of NASA, and Mrs. A. Bailie of AMA. The report also contains a description of the minimum variance digital program written by John Mohan of AMA based on the analysis contained herein. Portions of this program, relating to the computation of the nominal trajectory and the integration scheme, were taken from a FORTRAN program written by Miss E. Fisher of NASA.

TABLE OF CONTENTS

Notation

$\underset{\sim}{\epsilon}$	Lower bound of the error in a numerical solution
Δa	Difference operator, $\Delta a = a_{i+1} - a_i$
γ	Observation computable from the state
x	Vector of the state variables
ϵ	Observation noise
t	Time
A	Matrix of partial derivatives of the observations with respect to the initial state variables
A*	Transpose of **A**
$(\ )^{-1}$	Inverse of a matrix
W	Covariance matrix of the observations
P	Covariance matrix of the state variables
M	Matrix of the partial derivatives with respect to the instantaneous state variables
R	Position vector of vehicle relative to the dominant reference body
a	Semi-major axis
r	Scalar value of the vector **R**
v	Scalar value of the vector $\dot{\mathbf{R}}$
μ	Product of the universal gravitational constant and the mass of the dominant reference body
$\ddot{\mathbf{R}}$	Acceleration vector relative to the dominant reference body
P	Encke perturbation displacement vector
$\mathbf{R}_c$	Position vector of the local two-body orbit used in the Encke method
r_c	Scalar value of the vector $\mathbf{R}_c$
$\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$	Perturbation acceleration vectors defined in text
$\mathbf{R}_i$	Vector position of the vehicle from the i^{th} perturbing body

R_{ci}	Position vector of the i^{th} mass with respect to the reference body
J_2, J_3, J_4	Oblateness coefficients of the earth
x, y, z	Rectangular cartesian coordinates of the vector R
$\frac{A\ C_D}{m}$	Drag coefficient of the vehicle
ρ	Mass density of the atmosphere
Ω	Angular velocity vector of the dominant reference body; also right ascension of the ascending node
X	Vector cross product
R_r, $\dot{R}_r$	Initial position and velocity vectors of the osculating two-body orbit
f, g, f_t, g_t	Functions defined in text
r_r	Scalar value of R_r
v_c, v_r	Scalar values of $\dot{R}_c$, $\dot{R}_r$
d	Vector dot product, $R \cdot \dot{R}$
n	Mean motion
t_r	Time of rectification
f_1, f_2, f_3, f_4	Trigonometric functions defined in text
θ	Incremental eccentric anomaly
$\tilde{x}$	Random deviation of the variable x from its mean value
E (x)	Expected value of the variable x
ϵ_γ	Vector of instrument errors assumed to be uncorrelated with the position errors
$\bar{\epsilon}^2$	Covariance matrix of the observation instrument errors
K	Kalman filter
ϵ_x	Vector error of the state vector defined in text
δK	Variation in the operator K

$\dot{x}$	Time derivative of the vector x
α	Orbit parameter
$\Phi (t, t_o)$	Transition matrix of the state variables
Y	Covariance matrix of the estimated errors in the observations
t^+, t^-	The instant of time immediately following t and preceding t respectively
H	The angular momentum vector $H = R \times \dot{R}$ with components H_x, H_y, H_z
$R', \dot{R}'$	Fixed reference position and velocity vectors used to define small rotations about the instantaneous orbit vectors
S (t)	Point transformation matrix of the partial derivatives of the coordinates with respect to the parameters
$S^{-1}(t)$	Inverse point transformation matrix, relating the x's and the α's
$\Omega (t, t_o)$	Transition matrix of the variational parameters
β	General differentiating variable
$\alpha_i (t)$	The i^{th} parameter evaluated at time t
g, h, s, p	Functions defined in text
$\Psi (t, t_o)$	Approximating form of the parameter transition matrix
N (t)	Matrix of partial derivatives of the observations with respect to the parameters
L (t)	Modified Kalman filter
σ	Scalar range distance of the vehicle from a tracking station
x_s, y_s, z_s	Inertial geocentric coordinates of the tracking station
$\dot{\sigma}$	Range rate

ω_e	z component of the earth's siderial rotation rate
D	Declination angle
R A	Right ascension
x''', y''', z'''	Topocentric coordinates of the vehicle in the local horizon, local vertical system with x''' positive south, y''' positive east, and z''' positive upward along the local vertical
φ	Geodetic latitude
θ'	Right ascension of the tracking station meridian
A, E	Azimuth and elevation, respectively
l, m, n	Minitrack direction cosines

1. INTRODUCTION

The major requirements in generating a precision orbit prediction and tracking program arise from the following sources:

1. The need for obtaining a precise nominal trajectory which is capable of predicting the orbit position and velocity over long periods without loss of information due to round off.

2. A theoretically sound statistical process for obtaining information from observations rapidly without requiring large stretches of observations over long time arcs which unduly tax the linear assumption of the least square techniques.

3. The choice of variational parameters to be used for differential orbit corrections is of utmost importance in preventing the loss of information contained in observational data distributed over long time arcs.

4. The need for generating closed form analytical partial derivatives of the variations in the observations with respect to changes in the orbit parameters.

a) Orbit Prediction

The equations of motion of a vehicle in a gravitational field under the action of perturbative accelerations are highly nonlinear. Techniques for the solutions of these equations have occupied physicists and astronomers for centuries. Exact, closed form solutions are practically nonexistent. Even such an elementary problem as the restricted three-body problem has successfully eluded the capabilities of the best mathematicians. To date, the approximate methods which have been developed fall into two classes.

The first class, called special perturbation theories, is concerned with the numerical integration of the equations of motion. These methods are called special

because solutions so obtained are applicable only to the specific set of initial conditions used. Although these solutions are substantially correct, they suffer from the defect that the solution at time, t, cannot be obtained without prior integration through each value of time preceding the given time. These solutions require long computations and are subject to numerical round-off error due to integration.

The second class, called general perturbation methods, expresses the solution in approximate closed form by means of series, employing known algebraic and transcendental functions of time. The term general is derived from the fact that the solution is valid for any set of initial conditions and may be obtained by evaluating the solution for the given time. The difficulties encountered in these methods lie in the slow convergence and in the complexity of deriving the coefficients of the series.

The comparative advantages of the various special perturbation theories, such as Cowell's method, Encke's method, and the variation of parameters, have been extensively explored. The difficulties associated with the special perturbation method are well understood. The situation with regard to the general perturbation method is less satisfactory, and requires intensive investigation.

A brief discussion of the major sources of error in both general and special perturbation methods will be undertaken.

The major source of error in the general perturbation methods is the exclusion of small, though important, perturbative forces from the analysis. The great difficulty in expressing the disturbing function in terms of canonical variables and integrating the resulting equations often compels the exclusion of such effects. Another more fundamental error lies in the unavoidable growth of numerical accuracy due to the increasing magnitude of the independent variable, time.

It is a mistake to infer that simply because a solution is theoretically expressible in closed analytic form as a function of time, that it follows that

the numerical evaluation of this expression results in an accurate solution. To illustrate this fallacy, consider the function, sin t.

Since the function, sin t, can never be evaluated more accurately than its argument, t, the error in the computation of the function grows in direct proportion to the error contained in computing t. Since the computation takes place in a finite digit arithmetic on a machine, the increase in the magnitude of the independent variable is dropped as it grows beyond the capacity of the storage. This drop-off of the least significant portion of the function, t, is irretrievably lost to the digital program. Thus, if we have a finite digit arithmetic containing p digits, and compute in a number system, modulo s, the lower bound of the error in sin t is given by

$$\underline{\epsilon}\ (\sin t) = t s^{-p} \tag{1.1}$$

One of the most significant conclusions to be drawn from the above error estimate is the realization that it may be possible to obtain the same level of accuracy with a special perturbation numerical integration technique. In this manner it may be possible, on a finite digit arithmetic computing machine, (which after all is the only real method of computing in this world) to realize no numerical difference between the most accurate general perturbation closed form solution and a corresponding, well conceived, special perturbation method. In the sense of the error estimates given above, one may state that it is possible to eliminate round-off using numerical integration. What is really meant is that since the so-called exact solution has a built-in unavoidable error, the round-off error due to numerical integration can be so controlled as to produce an error no greater than the "exact" solution.

A study of the various special perturbation methods in use today for precision orbit prediction for the effect of round-off and machine time solution rates has been carried out in Reference 1. The general conclusions drawn in this study may be summarized as follows:

1. Cowell's method, while simple to program, consumes larger machine computed times and is subject to an unavoidable accumulation of the round-off error leading to the loss of orbit prediction accuracy.

2. The elimination of both truncation and round-off error can be accomplished through either the variation of parameters or the Encke method.

3. The preference of the Encke method over the variation of parameters arises from the simplicity of the equations and a great reduction in computing time. It is possible to generate a precision program using the Encke method which will produce a solution of the equations of motion as precisely as required in shorter computing time than any other available method.

As is well known, the Encke method solves the best local two-body problem and integrates the deviation from this nominal trajectory. Since the round-off error occurs only in the integrated position, it is possible to eliminate this defect from the answer by periodic orbit rectification. Additional difficulties of the conventional Encke method, such as numerical inaccuracies for circular orbits, etc., may be eliminated by using a solution of the Kepler problem in terms of the initial position and velocity vectors.

b) The Statistical Filter

Since the orbit position and velocity are not directly observable, it is necessary to infer these variables from a sequence of observations which are functions of the trajectory. In the conventional methods, a linear relationship is assumed between the deviations in the observations and the corresponding deviations in the orbit variables. Thus, an error in the orbit position will correspond to some predictable error in the observation. A large number of observations are made, overdetermining the linear system of equations. A least square technique is used to obtain the best value of the orbit errors to fit the known observation errors. Since the equations of motion are essentially nonlinear, this region of linearity becomes more and more constrictive about the nominal trajectory the longer the time period over which the prediction is made. Thus, the least square

technique often produces a result, fitting data over a long time arc, which is outside the linear range. This produces problems in convergence and consumes machine time. Reducing the number of observations to a shorter time arc helps avoid this difficulty. The weighted least squares is often used in this manner. However, a large number of observations is always needed in order to properly evaluate the effect of the random instrument errors.

The Schmidt-Kalman (References 2, 3) minimum variance technique avoids the difficulties mentioned above. This procedure permits a complete optimum estimate of the orbit variables and the observation errors from each single observation. It has been shown that the two methods converge to the same answer eventually for the same total set of data (Reference 4). Moreover, the variance technique always converges more rapidly since it requires less data at each stage than the least square procedure. At best, the least square technique can be said to be as good as the minimum variance.

It is instructive to contrast the iterative solution process of the three smoothing techniques described above.

Given the linear relation between observations and the required orbit information,

$$\Delta\gamma(t) = \sum \frac{\partial \gamma(t)}{\partial x(t_o)} \Delta x(t_o) + \epsilon , \qquad (1.2)$$

the following solutions obtain:

a) least squares

$$\Delta x(t_o) = (A^* A)^{-1} A^* \Delta\gamma(t) \qquad (1.3)$$

b) weighted least squares

$$\Delta x(t_o) = (A^* W^{-1} A + P^{-1})^{-1} A^* W^{-1} \Delta\gamma(t) \qquad (1.4)$$

c) Kalman filter

$$\Delta x(t) = P M^{*}(M P M^{*} + W)^{-1} \Delta \gamma (t) \tag{1.5}$$

The essential difference between the method of minimum variance, least squares and weighted least squares is contained in a comparison of Eqs. (1.3), (1.4) and (1.5).

The method of least squares and weighted least squares both relate the estimate of the initial parameters to an entire sequence of observational residuals spread over an extended time arc. In contrast, the method of minimum variance relates the present estimate of the state variable deviation to the present actual deviations in the observations. The linear assumptions required for the updating theory are violated to a much less degree in the method of minimum variance than in the method of weighted least squares.

c) <u>Choice of Parameters</u>

The choice of the elements used in the differential correction scheme is of utmost importance in predicting observations and other orbit functions over a long time period. It may be shown (Reference 5) that the best choice for parameters is that in which only one variable affects the energy or the mean motion of the orbit. The conventional astronomical elements have this property. However, three of these variables, the argument of perigee, the time of perigee passage, and the ascending node become poorly defined for near circular and low inclination orbits. The initial position and velocity components do not have this difficulty. However, all six of these affect the energy. An alternate set of elements in common use in tracking schemes (Reference 6) utilizes the scalar orbit distance (r) and velocity (v) in addition to four angles to describe the motion. For this set, two of the six variables affect the energy.

A convenient set of orbit parameters have been derived in Reference 7. These avoid the difficulties for circular and low inclination orbits as well as restricting the energy parameters to a single element. The parameters are given as follows:

1. A rigid rotation about the initial velocity vector.

2. A rigid rotation about the initial position vector.

3. A rigid rotation about the initial angular momentum vector.

4. A change in the variable, $\dfrac{R \cdot \dot{R}}{\sqrt{\mu |a|}}$.

5. A change in the reciprocal of the semi-major axis, $\dfrac{1}{a}$.

6. A change in the variable, $1 - \dfrac{r}{a}$.

These six elements have the characteristic that they determine the orbit independently of its orientation or shape and do not break down. Moreover, the matrix of partial derivatives of these elements contains only one secular term, namely that due to the semi-major axis, a.

d) Closed Form Analytical Derivatives

The requirement for utilizing closed form analytical derivatives is associated with the need for rapid computing times. If the program were required to integrate the variations in the observations due to changes in the orbit parameters directly from the differential equations for these variations, the computing time over and above that required for the nominal trajectory would increase by a factor of six (6). Since these variations are only required in order to obtain small iterative changes to the orbit parameters, approximate expressions will be useable, provided the residual in the observations can be accurately computed. This situation is analogous to the possible use of an approximate derivative in Newton's method for obtaining the roots of a polynomial. The program presented in this report derives a set of analytical derivatives based on the two-body problem approximation of the osculating orbit given in terms of the parameters outlined earlier. In a manner similar to the Encke method, a readjustment is made in the partial derivatives whenever the orbit is rectified.

2. PRECISION NOMINAL TRAJECTORY

a) The Equations of Motion

The equations of motion of a vehicle with negligible mass under the action of a dominant central force field and perturbed by other smaller forces is given by:

$$\ddot{R} = -\frac{\mu}{r^3} R + F_1 + F_2 + F_3 . \tag{2.1}$$

These equations may be written in the Encke form by replacing the vector R by the sum of a local two-body orbit position vector, R_c, plus a perturbation displacement,

$$R = R_c + P . \tag{2.2}$$

The vector R_c satisfies the differential equation,

$$\ddot{R}_c = -\mu \frac{R_c}{r_c^3} . \tag{2.3}$$

The Encke equation of motion for the perturbation displacement, P, is given by:

$$\ddot{P} = -\mu \left(\frac{R}{r^3} - \frac{R_c}{r_c^3} \right) + F_1 + F_2 + F_3 . \tag{2.4}$$

These are the equations that will be integrated to obtain a precision nominal trajectory.

The perturbations that are included in this program are those due to the gravitational attraction of the sun, moon, Venus, Mars, and Jupiter (F_1); the program also includes the perturbations due to the earth's oblateness (F_2) and the perturbations due to atmospheric drag (F_3).

The equations for the gravitational perturbation acceleration due to the sun, moon and planets are given by:

$$F_1 = - \sum_{i=1}^{5} \mu_i \left(\frac{R_i}{r_i^3} - \frac{R_{ci}}{r_{ci}^3} \right) \tag{2.5}$$

The perturbation accelerations due to the earth's oblateness are given by:

$$F_2 = b R + c K \tag{2.6}$$

where

$$b = \mu \left[\frac{J_2}{r^5} \left(- 1 + 5 \left(\frac{z}{r}\right)^2 \right) + \frac{J_3}{r^6} \frac{5}{2} \left(3 \frac{z}{r} - 7 \frac{z^3}{r^3}\right) \right.$$

$$\left. + 15 \frac{J_4}{r^7} \left(- 1 + 14 \frac{z^2}{r^2} - 21 \frac{z^4}{r^4}\right) \right] \tag{2.7}$$

$$c = \mu z \left[- \frac{2J_2}{r^5} + \frac{J_3}{r^5 z} \left(- \frac{3}{2} + \frac{15}{2} \frac{z^2}{r^2}\right) + \frac{20 J_4}{r^7} \left(- 3 + 7 \frac{z^2}{r^2}\right) \right].$$

The perturbations due to atmospheric drag are given by:

$$F_3 = - \frac{1}{2} \rho \frac{A C_D}{m} (\dot{R} - \Omega \times R) \left| \dot{R} - \Omega \times R \right| . \tag{2.8}$$

The vector $\dot{R} - \Omega \times R$ is the velocity of the vehicle relative to the atmosphere rotating rigidly with the earth. The vector Ω is the earth rotation vector and contains only a component in the z direction. Its magnitude is given by the earth's siderial rotation rate.

b) <u>Two-Body Problem</u>

The vector position and velocity for a Kepler orbit may be written in terms of the initial position and velocity vectors as given in Reference 8.

$$R_c = f\,R_r + g\,\dot{R}_r$$
$$\dot{R}_c = f_t\,R_r + g_t\,\dot{R}_r\,. \tag{2.9}$$

The functions f and g can be expressed for both elliptic and hyperbolic orbits by:

$$f = -\frac{|a|}{r_r}\,f_2 + 1$$

$$g = -\frac{1}{n}\,f_1 + (t - t_r)$$

$$\frac{r_c}{|a|} = f_2 + \frac{r_r}{|a|}\,f_4 + \frac{d_r}{\sqrt{\mu\,|a|}}\,f_3$$

$$\dot{f} = -\sqrt{\frac{\mu}{|a|}}\;\frac{1}{r_r}\;\frac{|a|}{r_c}\,f_3 \tag{2.10}$$

$$\dot{g} = -\frac{|a|}{r_c}\,f_2 + 1$$

$$n(t - t_r) = f_1 + \frac{r_r}{|a|}\,f_3 + \frac{d_r}{\sqrt{u\,|a|}}\,f_2\,.$$

The functions f_1, f_2, f_3, f_4 are defined in terms of the incremental eccentric anomaly $\theta = E - E_r$

$$f_1(\theta) = \theta - \sin\theta$$

$$f_2(\theta) = 1 - \cos\theta$$

$$f_3(\theta) = \sin\theta = \theta - f_1(\theta) \qquad \text{(elliptic)} \tag{2.11}$$

$$f_4(\theta) = \cos\theta = 1 - f_2(\theta)\,.$$

For the hyperbolic case

$$
\begin{aligned}
f_1(\theta) &= \sinh\theta - \theta \\
f_2(\theta) &= \cosh\theta - 1 \\
f_3(\theta) &= \sinh\theta = \theta + f_1(\theta) \\
f_4(\theta) &= \cosh\theta = 1 + f_2(\theta)
\end{aligned}
\tag{2.11a}
$$

where

$$r_r = (R_r \cdot R_r)^{1/2}$$

$$d_r = R_r \cdot \dot{R}_r$$

$$v_r^2 = \dot{R}_r \cdot \dot{R}_r$$

$$a = \left(\frac{2}{r_r} - \frac{v_r^2}{\mu}\right)^{-1}$$

$$n^2 = \frac{\mu}{|a|^3} \quad .$$

c) <u>Integration and Rectification Control</u>

The Encke method reduces somewhat the relative advantages of one integration scheme over another insofar as numerical accuracy is concerned. The method is capable of using almost any integration scheme to obtain a precise solution. The major advantage to be gained in the choice of integration schemes lies in the choice of the maximum integration interval to minimize the total computing time required. The Encke method computes the solution of the equations of

motion as a sum of the exact function plus the integrated effect of the perturbations. Thus the solution may be kept as precise as the exact portion so long as the accumulated error in the integrated portion is kept from affecting the least significant digit of the exact term. By estimating the accumulated round-off error and the accumulated truncation error, in the integrated portion of the solution, and by rectifying the solution to a new osculating Kepler orbit, when ever the integrated error threatens to affect the least significant digit of the exact solution, the total solution may be kept as precise as the exact term can be computed.

The particular program outlined in this report uses a fourth order Runge-Kutta integration scheme to initialize a sixth order backward difference second sum Cowell integration formula. A constant step size is used in place of a variable integration interval. At pre-set points in the trajectory the optimum interval size is altered, based on previous numerical experience with these intervals.

The rectification feature outlined above, based on round-off error control, is presently not in the program. At present, rectification is triggered whenever the integrated portion of the solution is a fixed ratio of the exact two-body term. In effect, this controls the accumulation of round-off error.

The rectification control for switching reference bodies is triggered as a function of the relative scalar distances to the various attracting bodies. The radius of the sphere of influence of each body is pre-set in the program. Whenever the vehicle enters the sphere of influence of a body, the program rectifies the orbit, and recomputes the planetary coordinates so that the distances and velocities of the various bodies are measured from the new dominant reference body.

3. THE STATISTICAL FILTER

a) Definitions

The deviation of the state vector from its expected mean value is given by:

$$\tilde{x} = x - \bar{x} \; . \tag{3.1}$$

The average of this deviation over all estimates of the state for a given time, t, is taken to be zero.

$$E(\tilde{x}) = 0 \quad , \quad E(x) = E(\bar{x}) = \bar{x} \; . \tag{3.2}$$

The covariance matrix of the expected deviation is given by

$$E(\tilde{x}, \tilde{x}^*) = \frac{1}{n} \sum_{1}^{n} \tilde{x} \cdot \tilde{x}^* = P(t) \; . \tag{3.3}$$

It is assumed that a limit exists for large n and that the matrix is bounded. This limit is the covariance matrix of the expected errors in the state variables at time t.

In addition, there exists a known deterministic function of the state which is assumed to be observable, and which is subject to random disturbances due to statistical noise. The deviation of the observation from its mean value at a given time is given by

$$\tilde{\gamma} = \gamma - \bar{\gamma}(x) \; . \tag{3.4}$$

The average deviation over all observations at this given time is assumed to be zero.

$$E(\tilde{\gamma}) = 0 \quad , \quad E(\gamma) = E(\bar{\gamma}) = \bar{\gamma}(x) \; . \tag{3.5}$$

The covariance matrix of the square of the deviations is given by

$$E(\tilde{\gamma}, \tilde{\gamma}^*) = \frac{1}{n} \sum_{1}^{n} \tilde{\gamma} \cdot \tilde{\gamma}^* = \bar{\epsilon}^2 \tag{3.6}$$

The error in the observations due to the noise is taken to be uncorrelated with the error in the state.

$$E(\tilde{x}, \epsilon^*) = E(\epsilon, \tilde{x}^*) = 0 \ . \tag{3.7}$$

A linear relation between an actual observation and its nominal deterministic value is given by

$$\gamma - \bar{\gamma}(\bar{x}) = \sum \frac{\partial \bar{\gamma}(t)}{\partial \bar{x}(t)} (x - \bar{x}) + \epsilon_\gamma \ .$$

$$\Delta\gamma = M \, \Delta x + \epsilon_\gamma \ . \tag{3.8}$$

The covariance matrix of the expected error in the observations due to both the observational noise and errors in the state, is given by

$$E(\gamma - \bar{\gamma}, [\gamma - \bar{\gamma}]^*) = M P M^* + \bar{\epsilon}^2 \ . \tag{3.9}$$

As the error in the state becomes negligible, the covariance matrix, P, tend to zero and

$$E(\Delta\gamma, \Delta\gamma^*) = E(\epsilon_\gamma, \epsilon_\gamma^*) = \bar{\epsilon}^2 \ . \tag{3.10}$$

b) <u>The Minimum Variance Filter</u>

Given the linear relation between the observation errors and the errors in the state, it is required to find the optimal linear unbiased estimate of the correction in the state as a function of the errors in the observation.

$$\Delta x(t) = K(t)\,\Delta\gamma(t) \tag{3.11}$$

The criteria used shall be such that over all possible operators, K, find that operator which makes the covariance matrix of the errors in Eq. (3.11) a minimum. Let,

$$\epsilon_x = \Delta x - K\Delta\gamma \ . \tag{3.12}$$

The covariance matrix of the error, ϵ_x, is

$$E(\epsilon_x, \epsilon_x^*) = P - K\,E(\Delta\gamma, \Delta x^*) - E(\Delta x, \Delta\gamma^*)K^* + K\,E(\Delta\gamma, \Delta\gamma^*)K^* \ . \tag{3.13}$$

The minimum is obtained by the condition

$$(P M^* - K Y)\,\delta K^* = 0 \tag{3.14}$$

The optimum filter is given by

$$K = P M^* Y^{-1} \tag{3.15}$$

c) Propagation of the Error

The propagation of the errors in the orbit may be obtained from the equations of motion. Let,

$$\dot{x} = f(x, t) \ . \tag{3.16}$$

A solution may be obtained for a given set of initial parameters required to integrate the equations of motion

$$x(t) = x(\alpha, t) \ . \tag{3.17}$$

The variation from this nominal solution due to changes in the given parameters is

$$\Delta x(t) = \sum \frac{\partial x(t)}{\partial \alpha} \Delta \alpha (t_o)$$

$$= \Phi(t) \; \Delta\alpha (t_o) \; . \tag{3.18}$$

The matrix $\Phi(t)$ is called the state transition matrix and propagates the terministic deviation in the state as a function of the known deviation in the initial parameters. The covariance matrix of the orbit errors at a later time, t, can be obtained from the knowledge of the errors in the parameters, α_i. Thus

$$E(\Delta x, \Delta x^{*}) = P(t) = \Phi \, E(\Delta\alpha, \Delta\alpha^{*}) \, \Phi^{*} \; . \tag{3.19}$$

Similarly given any function of the state, $\gamma(x)$ such as an observation, the propagation of errors in this observation may be estimated from a knowledge of the errors in the state and the instrument errors in the observation.

$$E(\Delta\gamma, \Delta\gamma^{*}) = M \, P \, M^{*} + \bar{\epsilon}^{2}. \tag{3.20}$$

Following each observation the change in the covariance matrix of the state may be computed through equation (3.13) after substituting the optimal value of the filter K.

$$P(t^{+}) = P(t^{-}) - P(t^{-}) \, M^{*} \, Y^{-1}. \tag{3.21}$$

4. CHOICE OF THE VARIATIONAL PARAMETERS

a) The Secular Effect

Numerical experience with the state transition and covariance matrices in linear prediction techniques has shown that the state transition matrix is numerically poorly conditioned for the computation of its determinant and its inverse over long time arcs. In the usual theory the initial conditions are used as the parameters. It is possible to define an alternate set of state parameters related to the conventional state variables by a non-vanishing transformation $S(t)$. The deviations of the state variables are related to the deviations of the variational parameters by

$$\Delta x = S(t)\,\Delta\alpha(t) \tag{4.1}$$

The state transition matrix of the state variables is related to the state transition matrix of the variational parameters by

$$\Phi(t, t_o) = S(t)\,\Omega(t, t_o)\,S^{-1}(t_o) \ . \tag{4.2}$$

The determinant of the transition matrix is given by

$$|\Phi(t, t_o)| = |S(t)| \cdot |\Omega(t, t_o)| \cdot |S^{-1}(t_o)| \tag{4.3}$$

Since the transformation between the state variables and the parameters is a point transformation, no functional dependence on time is contained in the determinant of the matrix $S(t)$. If the determinant of the state transition matrix is some polynomial in time, the corresponding determinant for the variational parameters has the same functional form in time, differing at most by a constant. The determinant of the state transition matrix is an invariant function of time. The difficulty in computing the determinant arises from the condition that the elements of the state transition matrix for one set of parameters may contain more secular terms than necessary. Consequently, the computation

of the determinant relies heavily on the cancellation of numbers of large and equal magnitude.

From the Hamilton-Jacobi theory, it is well known that the energy and the time are conjugate variables. It may be shown that the partial derivatives of the state variables with respect to any variable affecting the energy will produce secular terms. Thus,

$$\frac{\partial x(t)}{\partial \alpha(t_o)} = a + b(t - t_o) \tag{4.4}$$

where a, b are bounded functions. The secular term coefficient, b, will be zero only if

$$\frac{\partial (\text{energy})}{\partial \alpha(t_o)} \equiv 0 \, . \tag{4.5}$$

Thus it is desirable to obtain a set of variational elements in which a minimum number of variables affect the energy. The optimum set is one in which the energy is one of the variables and the others are independent of it. Such a set will contain the minimum number of elements with secular terms and will reduce the numerical difficulties with the state transition matrix.

b) <u>The Variational Parameters</u>

Rigid rotations do not affect the energy. In contrast to the conventional elements which have three rigid rotations (i, Ω, ω) about a fixed coordinate system, it is possible to obtain three rigid rotations about the vectors $\dot{R}$, R and H which do not relate to any fixed coordinate system.

To rigidly rotate a vector L about a given vector R through an angle α, the resulting transformation is given by

$$L' = \frac{R \cdot L}{r^2}(1 - \cos\alpha)\, R + \cos\alpha \; L + \frac{\sin\alpha}{r} \, R \times L \, . \tag{4.6}$$

In the limit, for small rotations, we obtain three infinitesimal rotation parameters

$$\alpha_1 = \lim_{R' \to R} \frac{v}{h^2} H \cdot R'$$

$$\alpha_2 = - \lim_{\dot{R}' \to \dot{R}} \frac{r}{h^2} H \cdot \dot{R}' \qquad (4.7)$$

$$\alpha_3 = - \lim_{\dot{R}' \to \dot{R}} \frac{1}{v^2 h} (H \times \dot{R}) \cdot \dot{R}'$$

The remaining three parameters are

$$\alpha_4 = \frac{R \cdot \dot{R}}{\sqrt{\mu |a|}}$$

$$\alpha_5 = \frac{1}{a} \qquad (4.8)$$

$$\alpha_6 = 1 - \frac{r}{a}$$

These six parameters will be used to carry out the differential correction process.

c) <u>The Transition Matrix</u>

The six (6) parameters are known functions of the state given by Eqs. (4.7) and (4.8). The partial derivatives of these variables may be obtained as follows:

$$\frac{\partial \alpha_1(t)}{\partial \beta} = -\frac{v}{h^2} H \cdot \frac{\partial R}{\partial \beta}$$

$$\frac{\partial \alpha_2(t)}{\partial \beta} = \frac{r}{h^2} H \cdot \frac{\partial \dot{R}}{\partial \beta}$$

$$\frac{\partial \alpha_3(t)}{\partial \beta} = \frac{1}{v^2 h} (H \times \dot{R}) \cdot \frac{\partial \dot{R}}{\partial \beta}$$

$$\begin{aligned}\frac{\partial \alpha_4(t)}{\partial \beta} &= \frac{1}{\sqrt{\mu |a|}} \left(\dot{R} - a \frac{R \cdot \dot{R}}{r^3} R\right) \cdot \frac{\partial R}{\partial \beta} \\ &+ \frac{1}{\sqrt{\mu |a|}} \left(R - a \frac{R \cdot \dot{R}}{\mu} \dot{R}\right) \cdot \frac{\partial \dot{R}}{\partial \beta}\end{aligned} \tag{4.9}$$

$$\frac{\partial \alpha_5(t)}{\partial \beta} = -\frac{2}{r^3} R \cdot \frac{\partial R}{\partial \beta} - \frac{2}{\mu} \dot{R} \cdot \frac{\partial \dot{R}}{\partial \beta}$$

$$\frac{\partial \alpha_6(t)}{\partial \beta} = \frac{v^2}{\mu r} R \cdot \frac{\partial R}{\partial \beta} + \frac{2r}{\mu} \dot{R} \cdot \frac{\partial \dot{R}}{\partial \beta}$$

The inverse transformation, $(\frac{\partial \alpha}{\partial x})$, may now be obtained.

$$\Delta \alpha = \left(\frac{\partial \alpha}{\partial x}, \frac{\partial \alpha}{\partial \dot{x}}\right) \Delta x = S^{-1} \Delta x$$

$$
S^{-1} = \begin{bmatrix}
-\frac{v}{h^2} H & 0 \\
0 & \frac{r}{h^2} H \\
0 & \frac{H \times R}{h v^2} \\
\frac{H \times R}{a^2 n r^2} - \frac{a}{r^3} \alpha_4 \alpha_6 R & -\frac{H \times \dot{R}}{a^2 n v^2} - \frac{2a}{r v^2} \alpha_4 \alpha_6 \dot{R} \\
-\frac{2R}{r^3} & \frac{2a}{\mu} \dot{R} \\
\frac{v^2}{\mu r} R & \frac{2r}{\mu} \dot{R}
\end{bmatrix} \tag{4.10}
$$

By choosing $\frac{1}{a}$ as a parameter, the other five parameters will automatically be independent of the energy providing the inverse of the matrix $(\frac{\partial \alpha}{\partial x})$ exists. This is guaranteed by defining the transformation matrix $S(t)$ such that

$$
(S^{-1}) S = I \tag{4.11}
$$

The point transformation matrix $S(t)$ is given by

$$
\Delta x = \begin{bmatrix} \frac{\partial x}{\partial \alpha} \\ \\ \frac{\partial \dot{x}}{\partial \alpha} \end{bmatrix} \Delta \alpha = S \, \Delta \alpha
$$

$$
S = \begin{bmatrix}
-\frac{H}{v} & 0 & \frac{H \times R}{h} & \frac{a^2 n}{h^2} H \times R & -aR & -\frac{a}{r} R + \frac{\mu^2 \alpha_6 \alpha_4 |a|}{h^2 r^2 v^2 n a} H \times R \\
0 & \frac{H}{r} & \frac{H \times \dot{R}}{h} & 0 & \frac{a}{2} \dot{R} & \frac{\mu a}{r^2 v^2} \dot{R}
\end{bmatrix} \tag{4.12}
$$

5. EVALUATION OF THE PARTIAL DERIVATIVES AND COVARIANCE MATRICES

a) The Transition Matrix

The computation of the partial derivatives of the observations with respect to the variational parameters are required. These may be obtained by integrating additional trajectories and forming the differences using the secant method. However, the program recommended in this report obtains the matrices of partial derivatives analytically in terms of the local Encke two-body orbital coordinates. Thus, the complete orbit prediction and partial derivative matrices may be obtained in essentially the same computing time as that of the nominal trajectory.

The method of obtaining the state transition matrix is based on a generalization of an Encke method applied to linear prediction theory. It is assumed that the equations of motion may be decomposed into two factors

$$\dot{x} = g(x, t) + h(x, t) \tag{5.1}$$

where

$$g >> h \ . \tag{5.2}$$

It is further assumed that a closed form solution of the differential equations is known for the case where $h = 0$,

$$\dot{s} = g(s, t) \ . \tag{5.3}$$

Furthermore, the state transition matrix for the approximating solution is known in closed form

$$\Delta s(t) = \Psi(t, t_o) \Delta s(t_o) \tag{5.4}$$

Let the deviation between the state variable and its approximation be given by

$$p(t) = x(t) - s(t) . \tag{5.5}$$

The perturbation equations of motion may now be written in the generalized Encke form

$$\dot{p} = g(x, t) - g(s, t) + h(x, t) . \tag{5.6}$$

In order to guarantee that the deviation, p, is never permitted to grow too large, the process of rectification is introduced. Whenever a pre-determined value of p is exceeded, the integration is terminated at time t_r. A new set of initial conditions are introduced, setting $p(t_r)$ equal to zero. Integration proceeds again about this new nominal approximate solution.

Since the deviation between x and s is never permitted to exceed the given value, the partial derivatives of the state variables from their nominal value may also be limited. Thus it is possible to write an approximate state transition matrix

$$\Omega(t, t_o) \cong \Psi(t, t_o) \quad \text{for } t \leq t_r . \tag{5.7}$$

Moreover, the approximate state transition matrix is known in closed form. Following each rectification, it is necessary to relate the state transition matrix at time t to the initial time. This may be accomplished by multiplying the approximate state transition matrix for times within each rectification interval by its value at the last rectification.

To obtain the transition matrix of the variational parameters, we note that

$$\frac{\partial\alpha(t)}{\partial\alpha(t_o)} = \sum \frac{\partial\alpha(t)}{\partial x(t)} \frac{\partial x(t)}{\partial x(t_o)} \frac{\partial x(t_o)}{\partial\alpha(t_o)} \tag{5.8}$$

In matrix form, this becomes

$$\Phi(t, t_o) = S^{-1}(t)\ \Psi(t, t_o)\ S(t_o)\ . \tag{5.9}$$

In this manner, a form of the transition matrix may be obtained which does not violate the condition of energy dependence. The matrix $\Phi(t, t_o)$ of the variational parameters is now given in closed form and only one element affects the energy.

b) The Parameter Transition Matrix

The parameter transition matrix, $\Psi(t, t_r)$ is given below:

$$\Delta\alpha(t) = \Psi\,\Delta\alpha(t_r)$$

$$\begin{matrix}
f\frac{v}{v_r} & -g\frac{v}{r_r} & 0 & 0 & 0 & 0 \\
-f_t\frac{r}{v_r} & g_t\frac{r}{r_r} & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & \ell_{34} & \frac{3}{2}\frac{\mu h a}{r^3 v^2}(t - t_r) & \ell_{36} \\
0 & 0 & 0 & g_t & \frac{3}{2}\frac{a^2 n}{r}\alpha_6(t - t_r) & -\frac{r_r}{a n} f_t \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -\frac{a n}{r} g & -\frac{3}{2}\frac{a^2 n}{r}\alpha_5(t - t_r) & \frac{r_r}{r} f
\end{matrix} \tag{5.10}$$

The terms ℓ_{34} and ℓ_{36} are given by

$$\ell_{34} = \frac{h}{v^2} a^2 n \left(\frac{g_t - 1}{r^2} - \frac{r_r f_t^2}{\mu} - \frac{f_t}{h^2} \left(\frac{\mu}{r} g - d_r \right) \right)$$

$$\ell_{36} = - \frac{h}{v^2} \frac{a}{r_r} f_t \left[\begin{array}{l} \frac{r_r}{r} f_4(\theta) + \frac{r_r^2}{r^2} - \frac{\mu g_t}{r_r v_r^2} \\ \\ + \frac{\mu d_r}{h^2 r_r v_r^2} \alpha_6(t_r) \left(\frac{\mu}{r} g - d_r \right) \end{array} \right] . \tag{5.11}$$

To obtain the parameter transition matrix over a time interval greater than one rectification interval, the following equation applies:

$$\Psi(t, t_o) = \Psi(t, t_r) \Psi(t_r, t_o) . \tag{5.12}$$

In this manner, a closed form expression for the state transition matrix may be obtained without integrating large quantities of differential equations. Moreover, the error in the computation may be limited by the proper use of the rectification technique.

As has been stated, the use of the conventional state variables, namely initial position and velocity, leads to a transition matrix which is poorly conditioned and which contains rapidly varying functions of time for its elements. In the procedure recommended here, the six variational parameters described above have only one secular term, namely that due to α_5. In addition they have the characteristic that they completely determine the orbit independently of the orbit orientation or shape.

c) The Modified Kalman Filter

A modification of the Kalman equations in terms of these new parameters has been derived and is available for incorporation in the orbit determination and prediction program.

The deviations of the orbit variables in terms of the new parameters are given by

$$\Delta x(t) = \left(\frac{\partial x}{\partial \alpha}\right) \Delta\alpha(t) = S(t)\,\Delta\alpha(t) . \qquad (5.13)$$

The parameter transition matrix is given by

$$\Delta\alpha(t) = \left(\frac{\partial\alpha(t)}{\partial\alpha(0)}\right) \Delta\alpha(0) = \Omega(t, t_o)\,\Delta\alpha(0) . \qquad (5.14)$$

The observation errors in terms of the new parameters are given by

$$\Delta\gamma(t) = N(t)\,\Delta\alpha(t) \qquad (5.15)$$

where

$$N(t) = M(t)\,S(t) .$$

The corresponding covariance matrices are given by

$$\begin{aligned} E(\Delta\alpha, \Delta\alpha^*) = Q(t) &= \Omega\, Q(0)\, \Omega^* \\ P(t) &= S\, Q(t)\, S^* \\ Y(t) &= N\, Q(t)\, N^* + \bar{\epsilon}^2 . \end{aligned} \qquad (5.16)$$

The inverse relationship between the orbit parameter corrections and the observational errors is given by

$$\Delta\alpha(t) = L(t)\,\Delta\gamma(t) \ . \tag{5.17}$$

The optimum filter $L(t)$ is given by

$$L(t) = Q\,N^{*}\,Y^{-1} \ . \tag{5.18}$$

The corrected covariance matrix after each observation is given by

$$Q(t^{+}) = Q(t^{-}) - Q(t^{-})\,N^{*}\,Y^{-1}\,N\,Q(t^{-}) \ . \tag{5.19}$$

Using these modified equations, it is now possible to use the Kalman scheme for both short and long term predictions. Moreover, the computing time need not exceed that necessary for a single nominal trajectory.

6. OBSERVATIONS

a) Types of Observations

One of the advantages of the Kalman scheme is the ability to determine the optimum observation to be made at each orbit position in order to have the greatest decrease in uncertainty. The method permits a predicted estimate of the decrease in uncertainty for each observation prior to have made it. In this manner, a choice may be made to obtain more rapid convergence to the proper solution. It is possible to use range and/or range rate at each position of the orbit in order to obtain the maximum information to be gained by using each or both. In addition, observations made from the vehicle during mid-course, from accelerometers, optical measurements, etc. may be used in an interspersed manner to optimum advantage.

The Kalman technique, by processing the data within the linear range of the prediction theory, and through the use of the recommended orbit correction parameters, makes it possible to process observations over large portions of the tracking complex.

The program will accept the following types of observational data, singly or in combination:

1. Range
2. Range rate
3. Right ascension and declination
4. Azimuth and elevation and Minitrack observations

b) Partial Derivatives of the Observations

In order to generate the differential corrections, it is necessary to compute residuals which consist of the difference between computed values of the observables and the observation data. In addition, it is necessary to compute partial

derivatives of the observables with respect to the orbit parameters. The range, range rate, right ascension and declination can be expressed directly in terms of the geocentric state variables and the required partial derivatives may be obtained as follows:

Range: The computed value of the range is given by

$$\sigma = \left[(x - x_s)^2 + (y - y_s)^2 + (z - z_s)^2 \right]^{1/2} \tag{6.1}$$

and the matrix of partial derivatives for the range is given by

$$M(t) = \left[\frac{x - x_s}{\sigma}, \frac{y - y_s}{\sigma}, \frac{z - z_s}{\sigma}, 0, 0, 0 \right] \tag{6.2}$$

Range Rate: The computed range rate is given by

$$\dot{\sigma} = \frac{1}{\sigma} \left[(x - x_s)(\dot{x} + \omega_e y_s) + (y - y_s)(\dot{y} - \omega_e x_s) + (z - z_s)\dot{z} \right] \tag{6.3}$$

The matrix of partial derivatives of the range rate with respect to the state variables is given by

$$\begin{aligned} M(t) = \Bigg[& \frac{\dot{x} + \omega_e y_s}{\sigma} - \dot{\sigma}\frac{x - x_s}{\sigma^2}, \frac{\dot{y} - \omega_e x_s}{\sigma} - \dot{\sigma}\frac{y - y_s}{\sigma^2}, \\ & \frac{\dot{z}}{\sigma} - \dot{\sigma}\frac{z - z_s}{\sigma^2}, \frac{x - x_s}{\sigma}, \frac{y - y_s}{\sigma}, \frac{z - z_s}{\sigma} \Bigg] \end{aligned} \tag{6.4}$$

Right Ascension and Declination: The expressions for the right ascension and the declination may be written as

$$\begin{aligned} \sin D &= \frac{z - z_s}{\sigma} \\ \tan RA &= \frac{y - y_s}{x - x_s} \end{aligned} \tag{6.5}$$

The matrix of partial derivatives for D is given by

$$M(t) = -\frac{z - z_s}{\sigma^2 \cos D} \left[\frac{x - x_s}{\sigma}, \frac{y - y_s}{\sigma}, \frac{(z - z_s)^2 - \sigma^2}{\sigma(z - z_s)}, 0, 0, 0\right] \tag{6.6}$$

and for RA by

$$M(t) = \frac{1}{\sec^2 RA} \left[-\frac{y - y_s}{(x - x_s)^2}, \frac{1}{x - x_s}, 0, 0, 0, 0\right] \tag{6.6a}$$

Azimuth and elevation and the Minitrack observations are most conveniently expressed in a topocentric, local horizon coordinate system and, to treat them, it is useful to introduce the following relation between the topocentric and geocentric coordinates:

$$\begin{matrix} x''' \\ y''' \\ z''' \end{matrix} = \begin{bmatrix} \sin\varphi\cos\theta' & \sin\varphi\sin\theta' & -\cos\varphi \\ -\sin\theta' & \cos\theta' & 0 \\ \cos\varphi\cos\theta' & \cos\varphi\sin\theta' & \sin\varphi \end{bmatrix} \begin{matrix} x - x_s \\ y - y_s \\ z - z_s \end{matrix} \tag{6.7}$$

This relation is used in developing the required partial derivatives for these angular observations.

Azimuth and Elevations: The expressions for azimuth and elevation are

$$A = \tan^{-1} \frac{y'''}{-x'''} \tag{6.8}$$

and

$$E = \tan^{-1} \frac{z'''}{(\sigma^2 - z'''^2)^{1/2}}$$

The corresponding matrices of partial derivatives are

For **A**

$$M(t) = -\frac{1}{\sigma^2 - z'''^2}\left[- x''' \sin\theta' - y''' \cos\theta' \sin\varphi,\right. \tag{6.9}$$

$$\left. x''' \cos\theta' - y''' \sin\theta' \sin\varphi,\ y''' \cos\varphi,\ 0,\ 0,\ 0\right]$$

and for **E**

$$M(t) = \frac{1}{(\sigma^2 - z'''^2)^{1/2}}\left[\cos\theta' \cos\varphi - \frac{z'''}{\sigma^2}(x - x_s),\right. \tag{6.9a}$$

$$\left. \sin\theta' \cos\varphi - \frac{z'''}{\sigma^2}(y - y_s),\ \sin\varphi - \frac{z'''}{\sigma^2}(z - z_s),\ 0,\ 0,\ 0\right]$$

Minitrack Observations: The Minitrack system direction cosines are expressed in terms of the topocentric coordinates as

$$\ell = -\frac{x'''}{\sigma}$$

$$m = \frac{y'''}{\sigma} \tag{6.10}$$

$$n = \frac{z'''}{\sigma}$$

The corresponding matrices of partial derivatives are

for ℓ

$$M(t) = \frac{1}{\sigma}\left[- \sin\varphi \cos\theta' + \frac{x'''(x - x_s)}{\sigma^2},\ - \sin\varphi \sin\theta' + \frac{x'''(y - y_s)}{\sigma^2},\right. \tag{6.11}$$

$$\left. \cos\varphi + \frac{x'''(z - z_s)}{\sigma^2},\ 0,\ 0,\ 0\right]$$

for m

$$M(t) = \frac{1}{\sigma}\left[- \sin\theta' - \frac{y'''(x - x_s)}{\sigma^2}, \cos\theta' - \frac{y'''(y - y_s)}{\sigma^2}, - \frac{y'''(z - z_s)}{\sigma^2}, 0, 0, 0 \right] \tag{6.11a}$$

and for n

$$M(t) = \frac{1}{\sigma}\left[\cos\varphi \cos\theta' - \frac{z'''(x - x_s)}{\sigma^2}, \cos\varphi \sin\theta' - \frac{z'''(y - y_s)}{\sigma^2}, \sin\varphi - \frac{z'''(z - z_s)}{\sigma^2}, 0, 0, 0 \right] \tag{6.11b}$$

c) Onboard Observations

The present program does not include onboard observations. It is recommended that in the extension for development of this program, this capability be added to the program.

7. PROGRAM INFORMATION

a) Program Operation

The program has two independent operational modes. The first mode concerns the generation of a reference orbit. The second mode involves the prediction of the orbit based upon observational data. Both modes require the planetary tables tape be placed on A-5.

Reference Orbit Mode

The purpose of this mode is to supply the state vectors and observation data that may be used in the prediction mode. The observations are output in the appropriate format as binary information on tape B-6. An option is available to generate an ephemeris of the reference orbit on tape B-8.

Prediction Mode

The observations for the prediction mode may be supplied either on cards or on logical tape # 16. An option is available to apply randomly distributed Gaussian noise to the data. When the data is exhausted the program will enter the reference orbit mode and continue in this mode until the final time is reached. When this occurs, separate options are available to obtain a summary of the residuals in the observations and the state variables. For the residuals in the observations, a scratch tape is needed on B-7. In order to obtain residuals in the state variables, the reference ephemeris is placed on B-9 and a scratch tape on B-8.

b) Input

The data is entered in eleven sections. Each section is preceded by a heading card with the section number entered in cols. 1-5 as an integer. Each piece of input will be defined as one of the four categories, integer, fixed point,

floating point or alphanumeric, by the notation, I, FX, FL or A. The quantity in the description column is entered on the specified card of the section in the appropriate columns. The name given is the name used for the quantity internally in the program.

Sect.	Card	Cols.	Name	Type	Description
1	1	2-72	ITITLE	A	Title
2	1	1-12	TIN	FL	Initial time, hr.
		13-24	TMAX	FL	Final time, hr.
		25-36	DTNE	FL	Integrating interval for near-earth portion of trajectory, hr.
		37-48	DTFE	FL	Integrating interval for far-earth portion of trajectory, hr.
		49-60	PRNTNE	FL	Print interval for near-earth, hr.
		61-72	PRNTFE	FL	Print interval for far-earth, hr.
	2	1-6	NYEARP	I	Year of trajectory
		7-12	DAYS	FX	Day of year
		13-18	HR	FX	Hr. of day
		19-24	HMIN	FX	Min. of hr.
		25-30	SEC	FX	Sec. of min.
	3	1-12	HMU	FL	Value of earth's gravitational constant, E. R. ^{3}/hr. ^{2}
		19-24	BMU	FX	6 values - represent planets - if planet is not used, insert 0 for that planet, if used insert 1. 1. Earth 2. Sun 3. Moon 4. Venus 5. Mars 6. Jupiter
3	1	1-5	MREF	I	Reference body (1-6)
		6-10	KLM	I	Indicator for dimension of input - R and R vectors may be input in 2 types of dimensions: 1. Earth radii, ER/hr. KLM = 0 2. km, km/sec. KLM = 1

Card	Cols.	Name	Type	Description
	11-15	NUMSTA	I	No. of stations input
	16-20	KDATA	I	Number of time points at which you have observation data; if the observation data is input on cards, it is not necessary to have the exact count.
	21-25	KPRTR	I	Set to non-zero to generate a time history of the trajectory on L. T. #18.
	26-30	IFLAG	I	Print control - set to non-zero for complete Kalman calculation.
	31-35	IOBS	I	Indicator to compute observations. 0 = No observation 1 = Observation 2 = Observation and summary (uses tape #17)
	36-40	LTBCD	I	BCD tape number for observation data.
	41-45	LTBIN	I	Binary tape number for observation data. 1. Computing observation of reference orbit = 16 2. Use Kalman scheme data on tape = 16 data on cards = 0
	46-50	KOND	I	Print indicator: set to 1 for additional print. If just summary is desired, use 0.
	51-55	INTPD	I	Indicator for Kalman scheme. INTPD = 2, process data all together at a time point. INTPD = 1, process each piece of data separately and pick best at each time point.
	56-60	NTEB	I	Number of $\bar{\epsilon}^{2}$ matrices (see Section 10)

Sect.	Card	Cols.	Name	Type	Description
4	1	1-12	CONJR	FL	1st harmonic coefficient of the earth's potential.
		13-24	CONAR	FL	2nd harmonic coefficient of the earth's potential.
		25-36	CONKR	FL	3rd harmonic coefficient of the earth's potential
		37-48	CDRAG	FL	Drag coefficient
		49-60	DCDRAG	FL	CDRAG increment
		61-72	AMASS	FL	Mass of vehicle
5	1	1-12	VNAME	A	Name of vehicle
		13-24	RMIN	FL	Minimum perigee distance
		25-36	TADD	FL	A large number should be used to generate Gaussian noise on a clean data tape. Otherwise use zero.
		37-48	CLUE	FL	Inhibitor for Kalman scheme
		49-58	ITYPE	I	Type of observation (see 7b)
6	1	1-12	PSI60	FL	Greenwich hour angle of 1960, rad.
		13-24	PDOT	FL	Daily rate of earth's rotation, rad./day
		25-36	PSIDOT	FL	Hourly rate of earth's rotation, rad./hr.
		37-48	ERAD	FL	Equatorial radius of earth, km.
		49-60	EPSSQ	FL	Ellipticity of earth
		61-72	AUERAD	FL	Astronomical unit
7	1	1-36	RCIN	FL	x, y, z (dimension determined by KLM)
		37-72	RDCIN	FL	$\dot{x}$, $\dot{y}$, $\dot{z}$ (dimension determined by KLM)
8	1 per sta.	1-2	K	I	Station number
		3-14	STANM	A	Station name
		15-26	SLON	FX	Longitude, deg.
		27-29	SLONM	FX	Longitude, min.
		30-35	SLONS	FX	Longitude, sec.
		36-47	SLAT	FX	Geodetic latitude, deg.
		48-50	SLATM	FX	Geodetic latitude, min.

Sect.	Card	Cols.	Name	Type	Description
		51-56	SLATS	FX	Geodetic latitude, sec.
		57-68	SALT	FX	Geodetic altitude, ft.
9	1-6	1-72	PMAT	FL	(6 x 6 matrix) initial estimate of the covariance matrix. Each card has one row of matrix.
10	1	12 per value	TIMEB	FL	Array of times associated with various $\bar{\epsilon}^2$ matrices. Time is time from epoch in hrs.
	1-4	12 per value	TEBAR	FL	$\bar{\epsilon}^2$ matrix - 4 x 4 error variance matrix, as many matrices as times; each card a row of matrix. Angles in seconds, range in meters, and range rate in cm/sec.
11	0				The section card is used to end the input data for each run.

c) Observation Data

The observation data may be supplied either on cards or on tape. If the data is input on cards, a tape is generated with the appropriate format which may be used for subsequent runs. The formats for the tape and cards are described below. It should be noted that the order of the observations can not be violated, i. e.,

1) Azimuth
2) Elevation
3) Range
4) Range rate
5) Right Ascension
6) Declination
7) open
8) open
9) open
10) open

The program is limited to ten types of observations.

Card Format

Since the format of observation cards vary with the source, the format itself is input on the first card. This is written exactly as a FORTRAN format statement except that the statement number and the word "Format" are eliminated. The program only requires that the information be ordered as follows:

1) Station number (as assigned in Section 8 of the input).
2) Time in days, hrs., mins., and seconds of the year.
3) Observations; blanks are used for missing observations.
   angles in degrees
   range in km.
   range rate in km./sec.
4) Type; see below.

A blank card follows the data.

Tape Format

A binary tape is supplied with the following information in each record.

1) Time from epoch in hrs., mins., and seconds.
2) Station number (as assigned in Section 8 of the input).
3) Type; see below.
4) Observations; blanks are not used for missing observations.
   angles in radians
   range in E. R.
   range rate in E. R./hr.

The observation type is specified by a sequence of ten values. Each value is associated with one of the ten types of observations in the reverse order of that given above.

A zero means the observation is not used and a one that it is used. The type may also be input in Section 5 of the input data. Its use here is to define the type to be generated when in the reference orbit mode. When in the prediction mode, a non-zero value is used to over-ride the type designation on the cards or tape.

d) Subroutine Description

SUBR.	EXPLANATION
ATANS	Computes arctangent in degrees
ATMSFR	Prepares atmospheric density table
CROSS	Computes cross product of two vectors
CWLAR	Computes either $\bar{P} = \bar{R} - \bar{R}_c$ and $\dot{\bar{P}} = \dot{\bar{R}} - \dot{\bar{R}}_c$ or $\bar{R} = \bar{R}_c + \bar{P}$ and $\dot{\bar{R}} = \dot{\bar{R}}_c + \dot{\bar{P}}$
DECHA	Sets up change in the integration interval
DEIN	Initializes the integration
DEREG	Normal integration routine
DERIV	Computes the derivatives
DERKI	Integrates by Runge-Kutta method
DOT	Computes dot product of two vectors
DRAG	Computes perturbations due to drag
FCOMP	Computes f_1, f_2, f_3, f_4 trig. functions
FIX	Sets up logic for data type
INT	Controls entries to integration routine
KEPLER	Computes two-body coordinates
LAG	Initializes positioning of the planetary tape
MATINV	Inverts a matrix
MATMPY	Performs minimum variance matrix operations
MDVECT	Computes $\|A\|$, $\|A\|^2$, $\|A\|^3$ of a vector $\bar{A}$
OBLATE	Computes perturbations due to oblateness
OBSER	Computes observations, i.e., range, range rate, azimuth, elevation, right ascension, and declination

SUBR.	EXPLANATION
OSCUL	Computes the osculating elements
PART	Computes $\Omega(t, t_o)$
PRINT	Controls output
RANUM	Generates random noise on data
RAPS	Computes perturbations due to radiation pressure
RCTTST	Tests for a rectification
READI	Generates positions of the planets
RECORD	Reads observation data
RECT	Performs a rectification
REDUCE	Reduces an angle to less than π
RPERG	Computes magnitude of the perigee vector
SMATRX	Computes S-matrix or its inverse
SUMARY	Summarizes the results of the minimum variance
VARORB	Initializes variation orbits
WORKMU	Prepares gravitational constants for the planets

8. RECOMMENDATIONS

The Kalman filter, as modified in this study, shows promise for application as both a real time and post flight tracking and orbit prediction program of extreme accuracy and computing speed. In order to realize these potential advantages, it is recommended that additional work and development be carried out along the following lines:

a) To determine in a systematic manner the effect on the final orbit information of

1. Rate of observation
2. Type of observation
3. Position in the orbit
4. Type of orbit.

b) Replace the formulation of the present two-body problem to permit the computation of parabolic and rectilinear cases. In particular, investigate the possibility of incorporating Herrick's formulae, if these are available.

c) Augment the program to include onboard observations of vehicle acceleration and star sightings as seen from the vehicle with respect to known geodetic and lunar sites.

d) Replace the present integration scheme with one which estimates the accumulated round-off and truncation errors for incorporation into the more rational rectification program.

e) Extend the modifications of the Kalman method to the case of a thrust vehicle. Obtain a new set of parameters and a state transition matrix for the thrusting case.

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